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— EDUCATION —

Mathematical Methods Examination 1

Solutions Book

VCE Trial Examination – Free

Contents pages

Question 1a (2 marks)

MARK 1. Applies product rule to find $f'(x)$
MARK 2. Provides correct answer

By the product rule, we have

$$f'(x) = \log_e(x) + 1 \implies f'(1) = 1.$$

Question 1b (2 marks)

MARK 1. Expresses integrand in useful form, or equivalent merit
MARK 2. Provides correct answer

From the formula sheet,

$$\int \frac{1}{\sqrt{1-x}}\,dx = \int (1-x)^{-\frac{1}{2}}\,dx = -\frac{1}{\frac{1}{2}}(1-x)^{\frac{1}{2}} + C = -2\sqrt{1-x} + C.$$

Question 2a (1 mark)

MARK 1. Applies factor theorem, or equivalent merit

Since $p(-2) = -8+8+6-6 = 0$, by the factor theorem, $x+2$ is a factor of $p(x)$.

Question 2b (2 marks)

MARK 1. Finds a quadratic factor of $p(x)$, or equivalent merit
MARK 2. Provides correct answer

For some real constants a, b and c, we have

$$p(x) = (x+2)\left(ax^2+bx+c\right) = ax^3 + (2a+b)x^2 + (2b+c)x + 2c.$$

Comparing coefficients, we must have $a = 1$, $b = 0$ and $c = -3$. Therefore,

$$p(x) = (x+2)\left(x^2-3\right) = 0 \implies x = -2,\ \pm\sqrt{3}.$$

Question 3a (2 marks)

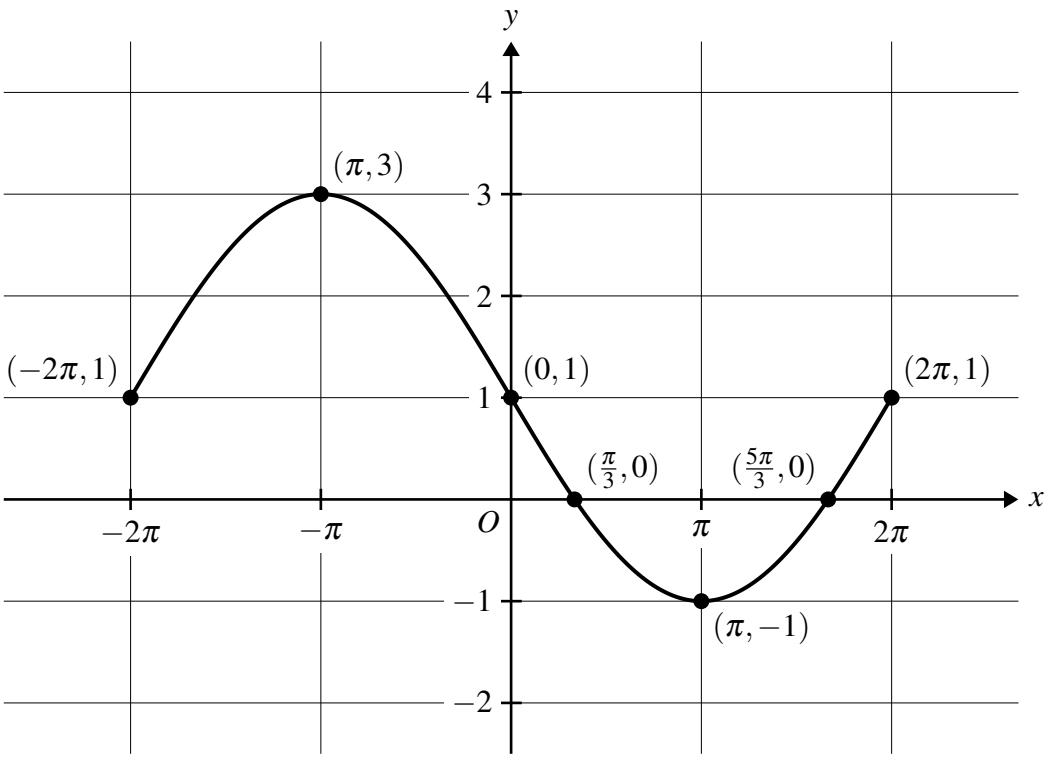
MARK 1. Finds relevant reference angle, or equivalent merit
MARK 2. Provides correct answer

Noting that $-\pi \leq \frac{x}{2} \leq \pi$, we get

$$\sin\left(\frac{x}{2}\right) = \frac{1}{2} \implies \frac{x}{2} = \frac{\pi}{6},\ \frac{5\pi}{6} \implies x = \frac{\pi}{3},\ \frac{5\pi}{3}.$$

Question 3b (3 marks)

MARK 1. Labels axial intercepts with coordinates
MARK 2. Labels endpoints and stationary points with coordinates
MARK 3. Shows correct graph shape



Question 4a (1 mark)

MARK 1. Provides correct answer

Using the formula for the area of a trapezium,

$$\Pr(1 \leq X \leq 3) = \frac{1}{2} \cdot 2 \cdot \left(\frac{1}{10} + \frac{3}{10} \right) = \frac{2}{5}.$$

Question 4b (3 marks)

MARK 1. Expresses expected value as integrals
MARK 2. Antidifferentiates integrands
MARK 3. Provides correct answer

The expected value of X is given by

$$\mathrm{E}(X) = \int_0^4 \frac{x^2}{10}\,dx + \int_4^5 \left(2x - \frac{2x^2}{5} \right) dx = \left[\frac{x^3}{30} \right]_0^4 + \left[x^2 - \frac{2x^3}{15} \right]_4^5.$$

Evaluating this gives

$$\mathrm{E}(X) = \frac{32}{15} - 0 + 25 - \frac{50}{3} - \left(16 - \frac{128}{15} \right) = 3.$$

Question 5a (2 marks)

MARK 1. Applies law of total probability
MARK 2. Provides correct answer

Let D be the event that the selected battery is defective. Let A and B be the events that it was produced by Machine A and Machine B respectively. Then, by the law of total probability,

$$\Pr(D) = \Pr(D \mid A)\Pr(A) + \Pr(D \mid B)\Pr(B) = \frac{1}{20} \cdot \frac{2}{5} + \frac{1}{10} \cdot \frac{3}{5} = \frac{2}{25}.$$

Question 5b (1 mark)

MARK 1. Provides correct answer

Using Bayes' theorem,

$$\Pr(B \mid D) = \frac{\Pr(D \mid B)\Pr(B)}{\Pr(D)} = \frac{\frac{1}{10} \cdot \frac{3}{5}}{\frac{2}{25}} = \frac{3}{4}.$$

Question 5c (1 mark)

MARK 1. Provides correct answer

Let $X \sim \text{Bi}\left(3, \frac{1}{10}\right)$. Then,

$$\Pr(X = 2) = \binom{3}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^1 = \frac{27}{1000}.$$

Question 6a (1 mark)

MARK 1. Provides correct answer

By the quotient rule,

$$f'(x) = \frac{-\pi x \sin(\pi x) - \cos(\pi x)}{x^2}.$$

Question 6b (2 marks)

MARK 1. Provides correct answer for tangent where $x = -1$
MARK 2. Provides correct answer for tangent where $x = 1$

The equation of the tangent where $x = -1$ is

$$y = f'(-1)(x+1) + f(-1) \implies y = x + 2.$$

The equation of the tangent where $x = 1$ is

$$y = f'(1)(x-1) + f(1) \implies y = x - 2.$$

Question 6c (2 marks)

MARK 1. Reasons valid method
MARK 2. Provides correct answer

A line perpendicular to these tangents is $y = -x$, which intersects the tangents at the points $(-1,1)$ and $(1,-1)$. The distance between the tangents is therefore

$$d = \sqrt{(-1-1)^2 + (1-(-1))^2} = 2\sqrt{2}.$$

Question 7a (1 mark)

MARK 1. Provides sufficient and correct reasoning

Since $f(x) \to 4$ as $x \to -\infty$, it follows that $a = 4$. Then,

$$0 = f(\log_e(16)) = 4 - 16^b \implies 4^{2b} = 4 \implies b = \frac{1}{2}.$$

Question 7b (2 marks)

MARK 1. Provides at least one correct transformation
MARK 2. Provides correct answer

The graph of f maps to the graph of g under the following transformations: a reflection in the x-axis, then a dilation by a factor of $\frac{1}{2}$ from the y-axis, followed by a translation of 2 in the positive y-direction.

Note: there are several possible answers.

Question 7c (4 marks)

MARK 1. Factorises $f(x) = g(x)$ as quadratic, or equivalent merit
MARK 2. Finds x-coordinate of intersection point
MARK 3. Antidifferentiates integrand of area integral, or equivalent merit
MARK 4. Provides correct answer

Observe that

$$f(x) = g(x) \implies 0 = e^x + e^{\frac{x}{2}} - 6 = \left(e^{\frac{x}{2}} + 3\right)\left(e^{\frac{x}{2}} - 2\right).$$

As $e^{\frac{x}{2}} > 0$ for all $x \in \mathbb{R}$, we get for the intersection point of the graphs

$$e^{\frac{x}{2}} = 2 \implies x = 2\log_e(2).$$

Therefore, the area of the shaded region is

$$A = \int_0^{2\log_e(2)} \left(f(x) - g(x)\right) dx = \int_0^{2\log_e(2)} \left(6 - e^{\frac{x}{2}} - e^x\right) dx = \left[6x - 2e^{\frac{x}{2}} - e^x\right]_0^{2\log_e(2)}$$

Evaluating this gives

$$A = 12\log_e(2) - 2\cdot 2 - 4 - (0 - 2 - 1) = 12\log_e(2) - 5.$$

Question 8a (1 mark)

MARK 1. Reasons valid method
MARK 2. Provides correct answer

The rule of the inverse of f is given by

$$x = \frac{f^{-1}(x)}{f^{-1}(x)+1} \implies xf^{-1}(x)+x = f^{-1}(x) \implies f^{-1}(x)(1-x) = x$$
$$\implies f^{-1}(x) = \frac{x}{1-x}.$$

Question 8b (1 mark)

MARK 1. Provides correct answer

The argument of the logarithm must be positive. That is, the numerator and denominator have the same sign. That is, $x > 0$ and $x+1 > 0$, or $x < 0$ and $x+1 < 0$. Hence, $\text{dom}(g) = (-\infty, -1) \cup (0, \infty)$.

Question 8c (2 marks)

MARK 1. Finds range of $f|_{\text{dom}(g)}$, or equivalent merit
MARK 2. Provides correct answer

We have that $f(x) = 1 - \frac{1}{x+1}$, and so $\text{ran}\left(f|_{\text{dom}(g)}\right) = (0,\infty) \setminus \{1\}$. Then since $\log_e$ is strictly increasing, it follows that $\text{ran}(g) = \mathbb{R} \setminus \{0\}$.

Question 8d (2 marks)

MARK 1. Invokes definition of inverse function, or equivalent merit
MARK 2. Provides complete proof of claim

We need to show that $\left((p \circ q) \circ \left(q^{-1} \circ p^{-1}\right)\right)(y) = y$ for all $y \in C$ and that $\left(\left(q^{-1} \circ p^{-1}\right) \circ (p \circ q)\right)(x) = x$ for all $x \in A$. Indeed, if $y \in C$, then

$$\left((p \circ q) \circ \left(q^{-1} \circ p^{-1}\right)\right)(y) = p\left(q\left(q^{-1}\left(p^{-1}(y)\right)\right)\right) = p\left(p^{-1}(y)\right) = y,$$

and if $x \in A$, then

$$\left(\left(q^{-1} \circ p^{-1}\right) \circ (p \circ q)\right)(x) = q^{-1}\left(p^{-1}\left(p\left(q(x)\right)\right)\right) = q^{-1}\left(q(x)\right) = x.$$

Hence, $(p \circ q)^{-1} = q^{-1} \circ p^{-1}$.

Question 8e (1 mark)

MARK 1. Provides correct answer

From **part d**, we have

$$g^{-1}(x) = \left(f|_{\text{dom}(g)}\right)^{-1}(e^x) = \frac{e^x}{1-e^x}.$$

End of Solutions Book

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